A MULTICHANNEL SCHEME IN SMOOTH SCATTERING THEORY

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ABSTRACT. We develop the scattering theory for a pair of self-adjoint operators $A_0 = A_1 \oplus \cdots \oplus A_N$ and $A = A_1 + \cdots + A_N$ under the assumption that all pair products $A_j A_k$ with $j \neq k$ satisfy certain regularity conditions. Roughly speaking, these conditions mean that the products $A_j A_k$, $j \neq k$, can be represented as integral operators with smooth kernels in the spectral representation of the operator A_0. We show that the absolutely continuous parts of the operators A_0 and A are unitarily equivalent. This yields a smooth version of Ismagilov's theorem known earlier in the trace class framework. We also prove that the singular continuous spectrum of the operator A is empty and that its eigenvalues may accumulate only to "thresholds" of the absolutely continuous spectra of the operators A_j. Our approach relies on a system of resolvent equations which can be considered as a generalization of Faddeev's equations for three particle quantum systems.

1. INTRODUCTION

1.1. **Trace class and smooth methods.** The methods used in scattering theory are naturally divided into two groups: trace class and smooth. The fundamental result of the trace class method is the Kato – Rosenblum theorem which establishes the existence of wave operators for a pair of self-adjoint operators A_0 and A under the assumption that the perturbation $A - A_0$ belongs to the trace class $\mathfrak{S}_1$. In particular, the absolutely continuous (a.c.) parts of A_0 and A turn out to be unitarily equivalent to each other.

D. Pearson extended the Kato – Rosenblum theorem to operators A_0 and A acting in different Hilbert spaces. More precisely, assuming that A_0 acts in $\mathcal{H}_0$, A acts in $\mathcal{H}$ and $J : \mathcal{H}_0 \to \mathcal{H}$ is a bounded operator ("identification") such that the effective perturbation $AJ - JA_0$ is trace class, Pearson proved that the wave operators (see definition (2.1) below) for the triple A_0, A, J exist. Earlier, sufficient conditions which ensure both[1] the existence and isometricity

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[1]Under Pearson's assumptions the wave operators are not necessarily isometric.

of wave operators in the two space setting were found by A. L. Belopol′skiĭ and M. Sh. Birman.

The smooth method in scattering theory (see, e.g., the original papers [3, 7] or the book [11]) is heavily based on the explicit spectral analysis of the unperturbed operator A_0. This approach requires that the perturbation, which we will denote by A_∞, be sufficiently "regular" (smooth) in the spectral representation of the operator A_0. This allows one to deduce information about the operator $A = A_0 + A_\infty$ from an equation relating the resolvents of the operators A_0 and A. Besides the existence and completeness of the wave operators for the pair A_0, A, the smooth method also yields nontrivial information about the singular component of the operator A. Normally, one proves that the singular continuous spectrum of A is empty and that the eigenvalues of A have finite multiplicities and can accumulate only to some exceptional spectral points (thresholds) of the operator A_0.

The standard scheme of smooth theory works for $\mathcal{H}_0 = \mathcal{H}$ and $J = I$. In the present paper we extend it (under rather special assumptions) to the two spaces framework.

1.2. **Ismagilov's theorem and its smooth version.** In [6], R. S. Ismagilov found an important generalization of the Kato – Rosenblum theorem. He considered the operator

$$A = A_\infty + A_1 + \cdots + A_N \tag{1.1}$$

where A_∞ and A_j are bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ and $A_\infty \in \mathfrak{S}_1$, $A_j A_k \in \mathfrak{S}_1$ for all $j \neq k$. The result of paper [6] can be formulated as follows. Consider the operator

$$A_0 = 0 \oplus A_1 \oplus \cdots \oplus A_N \tag{1.2}$$

in the Hilbert space

$$\mathcal{H}_0 = \mathcal{H}^{N+1} \stackrel{\text{def}}{=} \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{N+1}$$

i.e. $\mathcal{H}_0$ is the direct sum of $N+1$ copies of $\mathcal{H}$. Then the a.c. parts of the operators A_0 and A are unitarily equivalent to each other. The scattering theory for this pair A_0, A was constructed later in the papers [10] by A. V. Suslov and [5] by J. S. Howland and T. Kato where it was shown that the wave operators for A_0, A exist and are complete. Strictly speaking, in [6, 10, 5] the case $A_\infty = 0$, $N = 2$ was considered, but the general case reduces easily to this one.

Our goal is to develop the scattering theory for the operators A_0 and A given by (1.1) and (1.2) under the assumption that A_∞ and all pair products $A_j A_k$ with $j \neq k$ satisfy conditions of "smooth" type. In particular, we show that the a.c. parts of the operators A_0 and A are unitarily equivalent. This yields a smooth version of Ismagilov's theorem. We also prove that the singular

continuous spectrum of the operator A is empty and that its eigenvalues may accumulate only to the thresholds of the operators A_j.

The unitary equivalence of the a.c. parts of A_0, A, of course, means that the a.c. spectrum of A consists of the union of the a.c. spectra of $A_1,\ldots,$ A_N, and the a.c. subspace of A has the structure of the orthogonal sum of the subspaces corresponding to the operators A_j. In the terminology of the multiparticle scattering, each operator A_j contributes its own channel to the a.c. subspace of A. Because of this, we refer to the scattering problem (1.1), (1.2) for $N \geq 2$ as a *multichannel problem*, and call the case $N = 1$ the single channel case.

In the forthcoming paper [9] we apply our results to the spectral analysis of the operators of the type $\theta(H) - \theta(H_0)$, where H_0, H is a pair of operators for which the assumptions of the smooth scattering theory are satisfied, and θ is a piecewise continuous function. We are also planning to apply our results to the spectral theory of Hankel operators with piecewise continuous symbols θ. In the trace class framework, the latter problem was considered by J. S. Howland in [4]. In both cases, we represent the piecewise continuous function θ as a sum

$$\theta = \theta_\infty + \theta_1 + \cdots + \theta_N, \tag{1.3}$$

where θ_j are certain "standard" functions, each of which has only one discontinuity, and θ_∞ is a smooth function. Each θ_j gives rise to a "simple" explicitly diagonalisable self-adjoint operator A_j and thus (1.3) produces the decomposition (1.1) for $A = \theta(H) - \theta(H_0)$.

1.3. **A two spaces setup.** Actually, we consider the problem in a more general framework where self-adjoint operators A_0 and A act in different Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}$, respectively, and a bounded operator ("identification") $J : \mathcal{H}_0 \to \mathcal{H}$ is given. We do not assume J to be either isometric or invertible. For simplicity, we suppose that the operators A_0 and A are bounded. Under some assumptions, we construct a variant of smooth scattering theory for the triple A_0, A, J. This generalises the scheme well known for the case $\mathcal{H} = \mathcal{H}_0$, $J = I$. Our assumptions on the "abstract" triple A_0, A, J are rather special and are tailored in such a way that they are satisfied by the "concrete" operators (1.1), (1.2) and the identification

$$J : \mathcal{H}_0 \to \mathcal{H},$$
$$\mathbf{f} = (f_\infty, f_1, f_2, \ldots, f_N)^\top \mapsto J\mathbf{f} = f_\infty + f_1 + f_2 + \cdots + f_N. \tag{1.4}$$

Our main assumption on the "abstract" triple A_0, A, J is that the effective perturbation $AJ - JA_0$ can be factorised as

$$AJ - JA_0 = JT \tag{1.5}$$

where T is a bounded operator in $\mathcal{H}_0$ and the operator T^2 is compact. Moreover, the operator TA_0 (but not T itself) possesses some smoothness properties with respect to A_0.

For the triple (1.1), (1.2), (1.4), it can be easily checked by a direct inspection that factorisation (1.5) holds with the operator T given in the space $\mathcal{H}_0 = \mathcal{H}^{N+1}$ by the matrix

$$T = \begin{pmatrix} A_\infty & A_\infty & A_\infty & \dots & A_\infty \\ A_1 & 0 & A_1 & \dots & A_1 \\ A_2 & A_2 & 0 & \dots & A_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N & A_N & A_N & \dots & 0 \end{pmatrix}. \tag{1.6}$$

The importance of the factorisation (1.5) is as follows. Subtracting zJ from both sides, we get $(A-z)J = J(A_0-z) + JT$ for any $z \in \mathbb{C}$. From here, using the notation

$$R(z) = (A-z)^{-1}, \quad R_0(z) = (A_0-z)^{-1}, \tag{1.7}$$

we immediately obtain the resolvent identity

$$R(z)J(I + TR_0(z)) = JR_0(z), \quad \operatorname{Im} z \neq 0. \tag{1.8}$$

The key point of our construction is that we consider (1.8) as an equation for $R(z)J$ (rather than for $R(z)$). Under our assumptions, (1.8) turns out to be a Fredholm equation. Using this fact, we establish the limiting absorption principle for A. That is, we prove that $R(z)J$ has boundary values in an appropriate sense when z approaches the real axis staying away from the eigenvalues of A. Roughly speaking, our analysis hinges on inverting the operator $I + TR_0(z)$ in (1.8). Note that this operator acts in $\mathcal{H}_0$, and so we are able to carry out much of the analysis in the framework of a single Hilbert space $\mathcal{H}_0$ rather than in a pair of spaces $\mathcal{H}_0$, $\mathcal{H}$.

Analytic results obtained in this way allow us to verify the assumptions of smooth scattering theory for the operators A_0, A and the auxiliary identification $\widetilde{J} = JA_0$. Note that the identification $\widetilde{J}$ was used previously in [5] for the construction of the scattering theory for the pair (1.1), (1.2) under Ismagilov's assumptions. We work in the framework of a local version of scattering theory, that is, all our smoothness assumptions are made for the spectral parameter λ confined to an open bounded interval $\Delta \subset \mathbb{R}$ with $0 \notin \Delta$. We first construct local wave operators for the triple A_0, A, $\widetilde{J}$ and the interval Δ. Then using that $0 \notin \Delta$, we replace $\widetilde{J}$ by the original identification J. We show that the local wave operators for the triple A_0, A, J are isometric and complete. By the standard density arguments, global spectral results can be easily derived from the local ones, if necessary. We also show that the singular continuous

spectrum of the operator A is empty and that its eigenvalues in Δ do not have interior points of accumulation.

It is important that all our results for the "abstract" triple A_0, A, J apply to the "concrete" triple (1.1), (1.2), (1.4).

1.4. Comparison with a three particle scattering problem.
Let us discuss the analogy between (1.1) and the three particle Hamiltonian

$$H = H_0 + V_1 + V_2 + V_3 \quad \text{in } L^2(\mathbb{R}^{2d}), \, d \geq 1. \tag{1.9}$$

Here $H_0 = -\Delta$ is the operator of the total kinetic energy and V_j are potentials of interaction of pairs of particles (for example, V_1 corresponds to the interaction of the second and the third particles); the motion of the center of mass is removed. The potentials V_j do not decay at infinity, but the products $V_j V_k$ with $j \neq k$ possess this property. Thus, we have a formal analogy between (1.1) and (1.9) if $H_0 = 0$. In fact, this analogy goes further. Recall that every two particle Hamiltonian $H_j = H_0 + V_j$ yields its own channels of scattering to the three particle system (provided the corresponding two particle subsystem has a point spectrum). Similarly, in the problem (1.1), every operator A_j contributes its own band of the a.c. spectrum to the spectrum of the operator A. Furthermore, the resolvent equation (1.8) is algebraically similar to the famous Faddeev's equations [2] for the three particle quantum system. This is discussed in the Appendix.

Nevertheless, our problem preserves many features of the two particle scattering. Indeed, the two particle scattering matrix differs from the identity operator by a compact operator, and a result of this type remains true for the pair (1.1), (1.2). This should be compared with the fact that, as observed by R. Newton, the singularities of the three particle scattering matrix are determined by the scattering matrices for all two particle Hamiltonians H_j; see [12, Section 14.2] for a discussion of this phenomenon. In particular, the scattering matrix minus the identity operator is not compact in the three particle case.

1.5. The structure of the paper.
The basic definitions of scattering theory are formulated in the abstract framework in Section 2. Our main results concerning the pair (1.1), (1.2) are stated in Section 3 under Assumption 3.2. Section 4 plays the central role. Here we "forget" about the nature of the operators A_0, A, J, formulate a list of hypotheses (Assumption 4.1) on an "abstract" triple A_0, A, J and then develop a version of smooth scattering theory. In short Section 5, we show that under Assumption 3.2 these hypotheses are satisfied for the triple A_0, A and J defined by (1.1), (1.2) and (1.4). Then we translate the "abstract" results of Section 4 back into the setting of the operators A_j, A. In Section 6 we briefly discuss the stationary representations for the scattering matrix and the wave operators.

2. Scattering theory in a two spaces setting

In this preliminary section, we collect the required general definitions and results from scattering theory. For the details, see, e.g., the book [11].

2.1. **Wave operators.** Let A_0 (resp. A) be a bounded self-adjoint operator in a Hilbert space $\mathcal{H}_0$ (resp. $\mathcal{H}$). Throughout the paper, we denote by $E_0(\cdot)$ and $E(\cdot)$ the projection valued spectral measures of A_0 and A and by $P_0^{(\mathrm{ac})}$ and $P^{(\mathrm{ac})}$ the orthogonal projections onto the a.c. subspaces $\mathcal{H}_0^{(\mathrm{ac})}$ and $\mathcal{H}^{(\mathrm{ac})}$ of these operators, and set

$$E_0^{(\mathrm{ac})}(\Delta) = E_0(\Delta)P_0^{(\mathrm{ac})}, \quad E^{(\mathrm{ac})}(\Delta) = E(\Delta)P^{(\mathrm{ac})}$$

for $\Delta \subset \mathbb{R}$. We also use the notation $R_0(z)$, $R(z)$ for the resolvents of A_0, A, see (1.7). We denote by $\sigma_{\mathrm{ess}}(A)$ and $\sigma_p(A)$ the essential and point spectra of a self-adjoint operator A. The class of compact operators is denoted by $\mathfrak{S}_\infty$.

The wave operators for the operators A_0, A, a bounded operator $J : \mathcal{H}_0 \to \mathcal{H}$ and a bounded open interval Δ are defined by the relation

$$W_\pm(A, A_0; J, \Delta) = \operatorname*{s-lim}_{t \to \pm\infty} e^{iAt} J e^{-iA_0 t} E_0^{(\mathrm{ac})}(\Delta), \tag{2.1}$$

provided this strong limit exists. It is easy to see that under the assumption of its existence the wave operator possesses the intertwining property

$$W_\pm(A, A_0; J, \Delta)A_0 = AW_\pm(A, A_0; J, \Delta). \tag{2.2}$$

We need the following elementary assertion.

Lemma 2.1. *Suppose that the wave operator* (2.1) *exists. If for a real valued function* φ

$$J^*J - \varphi(A_0) \in \mathfrak{S}_\infty, \tag{2.3}$$

then

$$W_\pm^*(A, A_0; J, \Delta)W_\pm(A, A_0; J, \Delta) = \varphi(A_0)E_0^{(\mathrm{ac})}(\Delta). \tag{2.4}$$

In particular, if $J^*J - I \in \mathfrak{S}_\infty$*, then the operator* $W_\pm(A, A_0; J, \Delta)$ *is isometric on the subspace* $\operatorname{Ran} E_0^{(\mathrm{ac})}(\Delta)$.

Proof. Observe that

$$\|J e^{-iA_0 t} f\|^2 = ((J^*J - \varphi(A_0))e^{-iA_0 t} f, e^{-iA_0 t} f) + (\varphi(A_0)f, f). \tag{2.5}$$

Let $f \in \operatorname{Ran} E_0^{(\mathrm{ac})}(\Delta)$ and $t \to \pm\infty$. Then the l.h.s. of (2.5) tends to $\|W_\pm(A, A_0; J, \Delta)f\|^2$. By assumption (2.3), $\|(J^*J - \varphi(A_0))e^{-iA_0 t} f\| \to 0$ as $|t| \to \infty$, and hence the first term in the r.h.s. of (2.5) tends to zero as $|t| \to \infty$. Therefore passing in (2.5) to the limit $t \to \pm\infty$, we get (2.4). $\qquad\square$

The isometric wave operator $W_\pm(A, A_0; J, \Delta)$ is called complete if

$$\operatorname{Ran} W_\pm(A, A_0; J, \Delta) = \operatorname{Ran} E^{(\mathrm{ac})}(\Delta).$$

If both wave operators $W_\pm(A, A_0; J, \Delta)$ and $W_\pm(A_0, A; J^*, \Delta)$ exist, then they are adjoint to each other.

By the usual density arguments, if the local wave operators $W_\pm(A, A_0; J, \Delta_n)$ exist, are isometric and complete for a collection of intervals $\{\Delta_n\}$ such that $\mathbb{R} \setminus (\cup_n \Delta_n)$ has the Lebesgue measure zero, then the global wave operators $(\Delta = \mathbb{R})$ also exist, are isometric and complete. We state and prove all our results in the local setting, bearing in mind that the corresponding global results follow automatically.

2.2. **Relative smoothness.** Let us discuss sufficient conditions for the existence of the wave operators in the framework of the smooth scattering theory. Let A_0 be a self-adjoint operator in a Hilbert space $\mathcal{H}_0$, and let Q_0 be a bounded operator acting from $\mathcal{H}_0$ to another Hilbert space $\mathcal{K}$ (of course, the case $\mathcal{K} = \mathcal{H}_0$ is not excluded). One of the equivalent definitions of the A_0-smoothness of Q_0 on an interval δ in the sense of Kato is given by the condition

$$\sup_{\lambda \in \delta, \varepsilon \neq 0} \|Q_0(R_0(\lambda + i\varepsilon) - R_0(\lambda - i\varepsilon))Q_0^*\| < \infty, \quad R_0(z) = (A_0 - z)^{-1}.$$

Let us state the Kato-Lavine theorem.

Proposition 2.2. *Assume that the factorisation*

$$AJ - JA_0 = Q^*Q_0$$

with bounded operators $Q_0 : \mathcal{H}_0 \to \mathcal{K}$ and $Q : \mathcal{H} \to \mathcal{K}$ holds. Assume that the operators Q_0 and Q are smooth in the sense of Kato relative to A_0 and A, respectively, on every compact subinterval δ of an interval Δ. Then the wave operators $W_\pm(A, A_0; J, \Delta)$ and $W_\pm(A_0, A; J^*, \Delta)$ exist and are adjoint to each other.

Usually the smoothness of Q_0 with respect to the "unperturbed" operator A_0 can be verified directly. A proof of A-smoothness of the operator Q is more complicated. If $\mathcal{H}_0 = \mathcal{H}$, $J = I$ and $Q = BQ_0$ with a bounded operator B, then in applications the A-smoothness of Q is often deduced from a sufficiently strong form of A_0-smoothness of the operator Q_0 combined with some compactness arguments. This is discussed in in Subsection 2.3.

Let us formulate a convenient strong form of relative smoothness. Suppose that the spectrum of the operator A_0 on Δ is a.c. and has a constant (possibly infinite) multiplicity k. We consider a unitary mapping

$$F_0 : \operatorname{Ran} E_0(\Delta) \to L^2(\Delta; \mathfrak{h}_0) = L^2(\Delta) \otimes \mathfrak{h}_0, \quad \dim \mathfrak{h}_0 = k, \qquad (2.6)$$

of $\operatorname{Ran} E_0(\Delta)$ onto the space of vector-valued functions of $\lambda \in \Delta$ with values in an auxiliary Hilbert space $\mathfrak{h}_0$. Assume that F_0 maps A_0 to the operator of multiplication by λ, that is,

$$(F_0 A_0 f)(\lambda) = \lambda (F_0 f)(\lambda), \quad f \in \operatorname{Ran} E_0(\Delta). \tag{2.7}$$

We denote by $|\cdot|$ the norm in $\mathfrak{h}_0$. Note that

$$\frac{d(E_0(-\infty, \lambda)f, f)}{d\lambda} = |F_0(\lambda)f|^2 \tag{2.8}$$

for all $f \in \operatorname{Ran} E_0(\Delta)$ and almost all $\lambda \in \Delta$. Along with $L^2(\Delta; \mathfrak{h}_0)$ we consider the space $C^\gamma(\Delta; \mathfrak{h}_0)$, $\gamma \in (0, 1]$, of Hölder continuous vector-valued functions.

Definition 2.3. A bounded operator $Q_0 : \mathcal{H}_0 \to \mathcal{K}$ is called strongly A_0-smooth (with an exponent $\gamma \in (0, 1]$) on Δ if there exists a unitary diagonalization F_0 of $A_0|_{\operatorname{Ran} E_0(\Delta)}$ such that the operator $F_0 Q_0^*$ maps $\mathcal{K}$ continuously into $C^\gamma(\Delta; \mathfrak{h}_0)$, i.e.,

$$|(F_0 Q_0^* g)(\lambda)| \le C\|g\|, \quad |(F_0 Q_0^* g)(\lambda) - (F_0 Q_0^* g)(\mu)| \le C|\lambda - \mu|^\gamma \|g\|.$$

Here the constant C does not depend on λ and μ in compact subintervals of Δ.

For a strongly A_0-smooth operator Q_0, the operator $Z_0(\lambda; Q_0) : \mathcal{K} \to \mathfrak{h}_0$, defined by the relation

$$Z_0(\lambda; Q_0)g = (F_0 Q_0^* g)(\lambda), \tag{2.9}$$

is bounded and depends Hölder continuously on $\lambda \in \Delta$. According to (2.8) and (2.9) we have

$$\frac{d(Q_0 E_0(-\infty, \lambda) Q_0^* g, g)}{d\lambda} = |Z_0(\lambda; Q_0)g|^2 \tag{2.10}$$

so that, for a strongly A_0-smooth operator Q_0, this expression depends Hölder continuously on $\lambda \in \Delta$. Therefore the following result is a direct consequence (see, e.g., [11]) of the spectral theorem for A_0 and the Privalov lemma.

Proposition 2.4. *If an operator Q_0 is strongly A_0-smooth on Δ with some exponent $\gamma \in (0, 1)$, then the operator-valued function $Q_0 R_0(z) Q_0^*$ is Hölder continuous in the operator norm (with the same exponent γ) for $\operatorname{Re} z \in \Delta$ and $\pm \operatorname{Im} z \ge 0$.*

According to this result the strong A_0-smoothness of an operator Q_0 implies its A_0-smoothness in the sense of Kato on each compact subinterval δ of the interval Δ.

We will also need the following result (see, e.g., [11, Theorem 1.8.3]). It is known as the analytic Fredholm alternative.

Proposition 2.5. *Let $\mathcal{H}_0$ be a Hilbert space, and let Δ be an open interval. Suppose that the operator-valued function $G_0(z) : \mathcal{H}_0 \to \mathcal{H}_0$ defined on the set $\operatorname{Re} z \in \Delta$, $\operatorname{Im} z \neq 0$, is analytic, the operators $G_0(z)^p$ are compact for some natural p and the point -1 is not an eigenvalue of the operators $G_0(z)$. Assume also that $G_0(z)$ is continuous up to the cut along the real axis. Then $(I + G_0(z))^{-1}$ is a continuous operator-valued function of z for $\pm \operatorname{Im} z \geq 0$, $\operatorname{Re} z \in \Delta$, away from a closed set $\mathcal{N}_\pm \subset \Delta$ of measure zero. The set $\mathcal{N}_\pm$ consists of the points $\lambda \in \Delta$ where the equation*

$$g + G_0(\lambda \pm i0)g = 0$$

has a nontrivial solution $g \in \mathcal{H}_0$. Moreover, $(I + G_0(z))^{-1}$ is Hölder continuous if $G_0(z)$ is Hölder continuous.

2.3. **The "single channel" case.** Let us recall the well known basic results (see, e.g., [7] or [11, Sections 4.6, 4.7]) in the "single channel" setting (i.e., for $N = 1$ in (1.1)) as it provides a simple model for the multichannel case considered in the next section. Let A_0 and A_∞ be bounded self-adjoint operators in a Hilbert space $\mathcal{H}$ and let $A = A_0 + A_\infty$ (this is consistent with notation (1.1) if we set $N = 1$ and $A_1 = A_0$). Here A_0 is regarded as the "free" operator, A as the perturbed one, and A_∞ is the perturbation.

Proposition 2.6. *Let $\Delta \subset \mathbb{R}$ be a bounded open interval; assume that the spectrum of A_0 on Δ is purely a.c. with a constant multiplicity. Let Q_0 be a bounded operator with $\operatorname{Ker} Q_0 = \{0\}$; assume that Q_0 is strongly A_0-smooth on Δ with an exponent $\gamma > 1/2$. Assume that the operator A_∞ can be represented as $A_\infty = Q_0^* K Q_0$ with a compact operator K. Then the local wave operators*

$$W_\pm(A, A_0; \Delta) := \operatorname*{s-lim}_{t \to \pm\infty} e^{itA} e^{-itA_0} E_0(\Delta)$$

exist and enjoy the intertwining property (2.2). These operators are isometric on $\operatorname{Ran} E_0(\Delta)$ and are complete:

$$\operatorname{Ran} W_\pm(A, A_0; \Delta) = \operatorname{Ran} E^{(\mathrm{ac})}(\Delta).$$

The singular continuous spectrum of A on Δ is absent. All eigenvalues of A in Δ have finite multiplicities and can accumulate only to the endpoints of Δ.

Proposition 2.6, in particular, implies that the restriction of A onto $\operatorname{Ran} E^{(\mathrm{ac})}(\Delta)$ is unitarily equivalent to the restriction of A_0 onto $\operatorname{Ran} E_0(\Delta)$. That is, the a.c. spectrum of A on Δ has a constant multiplicity which coincides with the multiplicity of the a.c. spectrum of A_0 on Δ.

The key step in proof of Proposition 2.6 is the limiting absorption principle in the following form.

Proposition 2.7. *Under the hypotheses of Proposition 2.6 the operator valued function $Q_0 R(z) Q_0^*$ is Hölder continuous in the operator norm for $\operatorname{Re} z \in \Delta \setminus \sigma_p(A)$ and $\pm \operatorname{Im} z \geq 0$.*

The usual scheme of the proof of Proposition 2.7 proceeds as follows. First, one uses Proposition 2.4 to establish the Hölder continuity of $Q_0 R_0(z) Q_0^*$. Then, from the standard resolvent identity

$$R(z)(I + A_\infty R_0(z)) = R_0(z), \tag{2.11}$$

using the factorisation $A_\infty = Q_0^* K Q_0$, one obtains

$$Q_0 R(z) Q_0^*(I + K Q_0 R_0(z) Q_0^*) = Q_0 R_0(z) Q_0^*. \tag{2.12}$$

This is a Fredholm equation for the operator $Q_0 R(z) Q_0^*$. Applying now Proposition 2.5 to the operator valued function $G_0(z) = K Q_0 R_0(z) Q_0^*$, we see that $Q_0 R(z) Q_0^*$ is continuous for $\mathrm{Re}\, z \in \Delta$, $\pm \mathrm{Im}\, z \geq 0$ away from the exceptional set $\mathcal{N}_\pm$. It follows that the singular spectrum of the operator A is contained in the set $\mathcal{N}_+ \cap \mathcal{N}_-$. Under the assumption $\gamma > 1/2$ it is possible to prove that $\mathcal{N}_+ = \mathcal{N}_-$ and that this set consists of eigenvalues of A. Thus the singular continuous spectrum of the operator A in Δ is empty. Basically the same arguments show that the eigenvalues of A do not have in Δ interior points of accumulation.

It follows from Proposition 2.7 that the operator Q_0 is A-smooth in the sense of Kato on every compact subinterval of $\Delta \setminus \sigma_p(A)$. Thus Proposition 2.2 guarantees the existence of the wave operators $W_\pm(A, A_0; \Delta)$ and $W_\pm(A_0, A; \Delta)$. This ensures that the wave operators $W_\pm(A, A_0; \Delta)$ are isometric and complete.

Assumption $\gamma > 1/2$ in Proposition 2.6 is required only for the statements about the singular continuous and point spectra of A. Construction of the wave operators can be achieved under the weaker assumption $\gamma > 0$.

Note that, in the case $N = 1$, our resolvent equation (1.8) for the triple (1.1), (1.2), (1.4) reduces to (2.11).

3. Main results

3.1. **A gentle introduction: essential spectrum.** Weyl's theorem on the invariance of the essential spectrum of a self-adjoint operator under compact perturbations can be regarded as a precursor of scattering theory. Here we use this setting in order to illustrate the issues specific to our multichannel situation. The statement below is well known (see, e.g., the book [8]). We give the proof since it explains why the products $A_j A_k$ appear in the analysis of A. Our proof relies only on a direct construction of Weyl's sequences.

Proposition 3.1. *Let A_1, ..., A_N be bounded self-adjoint operators such that $A_j A_k \in \mathfrak{S}_\infty$ for $j \neq k$, and let the operator A be defined by formula (1.1). Then*

$$\sigma_{\mathrm{ess}}(A) = \bigcup_{j=1}^{N} \sigma_{\mathrm{ess}}(A_j) \quad (\mathrm{mod}\{0\}), \tag{3.1}$$

that is, the left and the right-hand sides coincide up to a possible zero eigenvalue of infinite multiplicity.

Of course, with notation (1.2), formula (3.1) can be equivalently rewritten as

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(A_0) \quad (\mathrm{mod}\{0\}).$$

Proof. If $\lambda \in \sigma_{\mathrm{ess}}(A_j)$ for some $j = 1, \ldots, N$ and $\lambda \neq 0$, then there exists a (Weyl) sequence f_n such that $\|f_n\| = 1$, $f_n \to 0$ weakly and $g_n := A_j f_n - \lambda f_n \to 0$ strongly as $n \to \infty$. Since the operators $A_k A_j$, $k \neq j$, are compact, it follows that for all $k \neq j$

$$A_k f_n = \lambda^{-1}(A_k A_j f_n - A_k g_n)$$

converge strongly to zero. Thus f_n is also the Weyl sequence for the operator A and the same λ.

Conversely, if $\lambda \in \sigma_{\mathrm{ess}}(A)$, then there exists a Weyl sequence f_n such that $\|f_n\| = 1$, $f_n \to 0$ weakly and $h_n := A f_n - \lambda f_n \to 0$ strongly as $n \to \infty$. Since the operators $A_k A_j$, $k \neq j$, are compact, this implies that, for all indices $j = 1, \ldots, N$, the sequences

$$(A_j - \lambda)A_j f_n = A_j(A - \sum_{k \neq j} A_k - \lambda)f_n = -\sum_{k \neq j} A_j A_k f_n + A_j h_n \to 0 \quad (3.2)$$

strongly as $n \to \infty$. Observe that if $\lambda \neq 0$, then at least for one of the indices $j = 1, \ldots, N$, the norm $\|A_j f_n\|$ does not tend to zero as $n \to \infty$. Indeed, supposing the contrary, we find that

$$\sum_{j=1}^{N} A_j f_n = A f_n = \lambda f_n + h_n$$

converges strongly to zero while the norm of the right-hand side tends to $|\lambda| \neq 0$. If $A_j f_n$ does not converge to zero, then we can assume that $\|A_j f_n\| \geq c > 0$ and set $\varphi_n = A_j f_n \|A_j f_n\|^{-1}$. It follows from (3.2) that $(A_j - \lambda)\varphi_n \to 0$ as $n \to \infty$. Moreover, $\varphi_n \to 0$ weakly because $f_n \to 0$ weakly as $n \to \infty$. $\square$

It is easy to see that the condition "mod $\{0\}$" cannot be dropped in (3.1). Indeed, let $\mathcal{H}$ be a Hilbert space of infinite dimension, and let $A_1 = \mathrm{diag}\{I, 0\}$, $A_2 = \mathrm{diag}\{0, I\}$ be diagonal operators in the space $\mathcal{H} \oplus \mathcal{H}$. Then $\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_2) = \{0, 1\}$ while $\sigma_{\mathrm{ess}}(A_1 + A_2) = \{1\}$.

3.2. **Multichannel scheme: main results.** Let $N \in \mathbb{N}$ and let A_∞, A_j, $j = 1, \ldots, N$, be bounded self-adjoint operators in a Hilbert space $\mathcal{H}$. As in Section 1, we set

$$A = A_\infty + A_1 + A_2 + \cdots + A_N.$$

Let $\Delta \subset \mathbb{R}$ be a bounded open interval with $0 \notin \Delta$, and let X be a bounded operator in $\mathcal{H}$. We need

Assumption 3.2.   (i) *One has*

$$\operatorname{Ker} X = \operatorname{Ker} X^* = \{0\}.$$

(ii) *The spectra of the operators A_1, ..., A_N in Δ are a.c. and have constant multiplicities. For all $j = 1, \ldots, N$, the operator X is strongly A_j-smooth (see Definition 2.3) on Δ with an exponent $\gamma > 1/2$.*

(iii) *The operator A_∞ can be represented as $A_\infty = X^* K_\infty X$ with a compact operator K_∞.*

(iv) *For all $j \neq \ell$, the operators $A_j A_\ell$ can be represented as*

$$A_j A_\ell = X^* K_{j,\ell} X \tag{3.3}$$

where the operators $K_{j,\ell}$ are compact.

(v) *The operators $X A_j X^{-1}$ are bounded[2] for all $j = 1, \ldots, N$.*

Our spectral results are collected in the following assertion.

Theorem 3.3. *Under Assumption* 3.2 *one has:*

(i) *The a.c. spectrum of A on Δ has a constant multiplicity equal to the sum of the multiplicities of the a.c. spectra of A_j, $j = 1, \ldots, N$, on Δ.*

(ii) *The singular continuous spectrum of A on Δ is empty. The eigenvalues of A in Δ have finite multiplicities and can accumulate only to the endpoints of Δ.*

(iii) *The operator valued function $X R(z) X^*$ is Hölder continuous (in the operator norm) in z for $\pm \operatorname{Im} z \geq 0$ and $\operatorname{Re} z \in \Delta \setminus \sigma_p(A)$.*

The scattering theory for the set of operators $A_1, \ldots, A_N$ and the operator A is described in the following assertion. We denote by $E_j(\Delta)$ the spectral projection of A_j corresponding to the interval Δ.

Theorem 3.4. *Under Assumption* 3.2 *one has:*

(i) *For all $j = 1, \ldots, N$, the local wave operators*

$$W_\pm(A, A_j; \Delta) = \operatorname*{s-lim}_{t \to \pm\infty} e^{iAt} e^{-iA_j t} E_j(\Delta)$$

exist and enjoy the intertwining property

$$A W_\pm(A, A_j; \Delta) = W_\pm(A, A_j; \Delta) A_j.$$

The operators $W_\pm(A, A_j; \Delta)$ are isometric on $\operatorname{Ran} E_j(\Delta)$. Their ranges are orthogonal to each other:

$$\operatorname{Ran} W_\pm(A, A_j; \Delta) \perp \operatorname{Ran} W_\pm(A, A_\ell; \Delta), \quad j \neq \ell. \tag{3.4}$$

---

[2]Strictly speaking, we assume that the operators $X A_j X^{-1}$ defined on the dense set $\operatorname{Ran} X$ extend to bounded operators. The same convention applies to all other operators of this type.

(ii) *The asymptotic completeness holds:*

$$\bigoplus_{j=1}^{N} \operatorname{Ran} W_{\pm}(A, A_j; \Delta) = \operatorname{Ran} E^{(\mathrm{ac})}(\Delta). \tag{3.5}$$

We note that the first statement of Theorem 3.3 is a direct consequence of Theorem 3.4.

As already mentioned in the Introduction, in the trace class framework (Ismagilov's theorem) the statements of Theorem 3.4 (but not parts (ii, iii) of Theorem 3.3) are obtained in [4, 10] under the assumptions $A_j A_k \in \mathfrak{S}_1$, $j \neq k$, and $A_\infty \in \mathfrak{S}_1$. Part (i) of Theorem 3.3 goes back to [6].

4. A TWO SPACES SETUP

4.1. **Assumptions and results.** Let A_0 (resp. A) be a bounded self-adjoint operator in a Hilbert space $\mathcal{H}_0$ (resp. $\mathcal{H}$). Let $J : \mathcal{H}_0 \to \mathcal{H}$ be a bounded operator (the "identification"). Our key assumption is that the factorisation (1.5) holds true with a bounded operator T in $\mathcal{H}_0$. We fix an open bounded interval $\Delta \subset \mathbb{R} \setminus \{0\}$ and a bounded operator Q_0 in $\mathcal{H}_0$. Below we present a suitable version of scattering theory for the triple (A_0, A, J) under the following

Assumption 4.1. (i) $\operatorname{Ker} J^* = \{0\}$.
  (ii) $\operatorname{Ker} J \cap \operatorname{Ker}(A_0 + T - z) = \{0\}$ *for all* $z \neq 0$.
 (iii) $\operatorname{Ker} Q_0 = \operatorname{Ker} Q_0^* = \{0\}$.
 (iv) *The spectrum of A_0 on Δ is purely a.c. with a constant multiplicity. The operator Q_0 in $\mathcal{H}_0$ is strongly A_0-smooth on Δ with an exponent $\gamma \in (1/2, 1)$.*
  (v) *The operator $T A_0$ can be factorised as*

$$T A_0 = Q_0^* K Q_0, \tag{4.1}$$

   where K is a compact operator in $\mathcal{H}_0$.
 (vi) *The operator $M_0 = Q_0 T^* Q_0^{-1}$ is bounded on $\mathcal{H}_0$ and the operator M_0^2 is compact.*
(vii) *The operator $M = Q_0 J^* J Q_0^{-1}$ is bounded.*
(viii) *One has*

$$A_0(J^* J - I) A_0 = Q_0^* \widetilde{K} Q_0$$

   with a compact operator $\widetilde{K}$ in $\mathcal{H}_0$.
 (ix) *The operator $J A_0^2 J^* - A^2$ is compact.*

We do not require that $\operatorname{Ker} J = \{0\}$; it turns out that instead of this it suffices to impose a much weaker Assumption 4.1(ii).

Our spectral results are formulated in the following assertion.

Theorem 4.2. *Under Assumption* 4.1 *one has:*

(i) *The a.c. spectrum of the operator A on Δ has the same multiplicity as that of the operator A_0.*
(ii) *The singular continuous spectrum of A on Δ is empty.*
(iii) *The eigenvalues of A in Δ have finite multiplicities and can accumulate only to the endpoints of Δ.*
(iv) *Let*
$$Q = Q_0 J^* : \mathcal{H} \to \mathcal{H}_0.$$
Then the operator valued function $QR(z)Q^*$ is Hölder continuous (in the operator norm) in z for $\pm\operatorname{Im} z \geq 0$ and $\operatorname{Re} z \in \Delta \setminus \sigma_p(A)$.

Scattering theory is described in the following assertion.

Theorem 4.3. *Under Assumption 4.1 one has:*

(i) *The wave operators $W_\pm(A, A_0; J, \Delta)$ exist and enjoy the intertwining property*
$$W_\pm(A, A_0; J, \Delta)A_0 = AW_\pm(A, A_0; J, \Delta).$$
(ii) *The operators $W_\pm(A, A_0; J, \Delta)$ are isometric on $\operatorname{Ran} E_0(\Delta)$ and are complete:*
$$W_\pm^*(A, A_0; J, \Delta)W_\pm(A, A_0; J, \Delta) = E_0(\Delta), \tag{4.2}$$
$$W_\pm(A, A_0; J, \Delta)W_\pm^*(A, A_0; J, \Delta) = E^{(\mathrm{ac})}(\Delta). \tag{4.3}$$

4.2. **Fredholm resolvent equations.** Let us proceed from the resolvent identity (1.8) which we consider as an equation for the operator $R(z)J$. This equation is Fredholm if
$$(TR_0(z))^p \in \mathfrak{S}_\infty \quad \text{for some } p \in \mathbb{N}. \tag{4.4}$$

We first consider the homogeneous equation corresponding to (1.8). The argument of the following lemma will be used several times in what follows.

Lemma 4.4. *Let Assumption 4.1(ii) hold. Then for any z, $\operatorname{Im} z \neq 0$, the equation*
$$f + TR_0(z)f = 0 \tag{4.5}$$
has only the trivial solution $f = 0$.

Proof. Set $\varphi = R_0(z)f$. It follows from (4.5) that
$$(A_0 - z)\varphi + T\varphi = 0. \tag{4.6}$$

Applying the operator J to this equation and using (1.5), we see that $AJ\varphi = zJ\varphi$. Hence $J\varphi = 0$ because the operator A is self-adjoint. Moreover, $\varphi \in \operatorname{Ker}(A_0 + T - z)$ according to equation (4.6). Therefore Assumption 4.1(ii) implies that $\varphi = 0$ whence $f = 0$. $\qquad\square$

In view of equation (1.8), this directly leads to the following assertion.

Lemma 4.5. *If Assumption* 4.1(ii) *and inclusion* (4.4) *are satisfied, then*

$$R(z)J = JR_0(z)(I + TR_0(z))^{-1}, \quad \text{Im}\, z \neq 0, \tag{4.7}$$

where the inverse operator in the r.h.s. exists and is bounded.

Equation (4.7) is convenient for $\text{Im}\, z \neq 0$. In order to study the resolvent $R(z)$ when z approaches the real axis, we need to sandwich this resolvent between the operators Q, Q^* (recall that $Q = Q_0 J^*$) and rearrange equation (1.8). This should be compared to passing from (2.11) to (2.12) in the "single channel" setting. In the present case, the algebra is slightly more complicated.

Lemma 4.6. *Let Assumption* 4.1(v–vii) *hold. Then for all* $\text{Im}\, z \neq 0$, *we have*

$$QR(z)Q^*(I + G_0(z)) = MQ_0R_0(z)Q_0^*, \tag{4.8}$$

where the operator $G_0(z)$ *is given by*[3]

$$G_0(z) = -z^{-1}M_0^* + z^{-1}KQ_0R_0(z)Q_0^* \tag{4.9}$$

and hence $G_0^2(z) \in \mathfrak{S}_\infty$.

Proof. First, we multiply (1.8) by Q on the left and by Q_0^* on the right:

$$QR(z)Q^* + QR(z)JTR_0(z)Q_0^* = QJR_0(z)Q_0^*. \tag{4.10}$$

Let us show that

$$TR_0(z)Q_0^* = Q_0^*G_0(z). \tag{4.11}$$

Indeed, using the identity

$$zR_0(z) = -I + A_0R_0(z), \tag{4.12}$$

we see that

$$zTR_0(z)Q_0^* = -TQ_0^* + (TA_0)R_0(z)Q_0^*. \tag{4.13}$$

By Assumption 4.1(vi), we have

$$TQ_0^* = Q_0^*M_0^*.$$

Using also equality (4.1), we can rewrite (4.13) as

$$TR_0(z)Q_0^* = z^{-1}Q_0^*(-M_0^* + KQ_0R_0(z)Q_0^*).$$

By definition (4.9), this yields (4.11).

Substituting (4.11) into the second term in the l.h.s. of (4.10) and using that $JQ_0^* = Q^*$, we see that this term equals $QR(z)Q^*G_0(z)$. In the r.h.s. of (4.10), we use the fact that, by Assumption 4.1(vii),

$$QJ = Q_0J^*J = MQ_0. \tag{4.14}$$

Therefore (4.10) yields identity (4.8).

The operator $G_0^2(z)$ is compact because, by Assumption 4.1(v,vi), both K and M_0^2 are compact. $\qquad\square$

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[3]Formally, $G_0(z) = (Q_0^*)^{-1}TR_0(z)Q_0^*$.

The resolvent equation (4.8) allows us to study the boundary values of $R(z)$ as $\operatorname{Im} z \to 0$. It should be compared to the resolvent equation (2.12) in the "single channel" setting. The main difference between the single channel resolvent equation and the multichannel one is that in the single channel case, the operator $KQ_0R_0(z)Q_0^*$ is compact, whereas under Assumption 4.1 we cannot guarantee the compactness of $G_0(z)$; instead, we have the compactness of the square $G_0^2(z)$. In any case, (4.8) is a Fredholm equation for $QR(z)Q^*$ amenable to analysis for $\operatorname{Im} z \to 0$.

Lemma 4.7. *Let the operator $G_0(z)$ be defined by formula* (4.9). *Then under Assumption* 4.1(ii, iii, v − vii), *the equation*

$$g + G_0(z)g = 0, \quad \operatorname{Im} z \neq 0, \tag{4.15}$$

has only the trivial solution $g = 0$.

Proof. Applying the operator Q_0^* to equation (4.15) and taking identity (4.11) into account, we obtain equation (4.5) for $f = Q_0^*g$. By Lemma 4.4 we have $f = 0$. This ensures that $g = 0$ because $\operatorname{Ker} Q_0^* = \{0\}$. □

Since $G_0^2(z) \in \mathfrak{S}_\infty$, Lemma 4.7 implies the following assertion.

Lemma 4.8. *Under the hypothesis of Lemma* 4.7, *we have the representation*

$$QR(z)Q^* = MQ_0R_0(z)Q_0^*(I + G_0(z))^{-1}, \quad \operatorname{Im} z \neq 0, \tag{4.16}$$

where the inverse operator in the r.h.s. exists and is bounded.

4.3. **The limiting absorption principle and spectral consequences.** Let us now study equation (4.16) as z approaches the interval Δ. The first assertion is a direct consequence of Proposition 2.4.

Lemma 4.9. *Under Assumption* 4.1(iv) *the operator valued functions $Q_0R_0(z)Q_0^*$ and hence $G_0(z)$ are Hölder continuous with exponent $\gamma \in (1/2, 1)$ for $\operatorname{Re} z \in \Delta$, $\pm\operatorname{Im} z \geq 0$.*

Let the set $\mathcal{N}_\pm \subset \Delta$ consist of the points λ such that the equation

$$g + G_0(\lambda \pm i0)g = 0 \tag{4.17}$$

has a nontrivial solution $g \neq 0$. By Proposition 2.5, the set $\mathcal{N}_\pm$ is closed and has the Lebesgue measure zero. The inverse operator in the r.h.s. of (4.16) is continuous for $\pm\operatorname{Im} z \geq 0$ away from the set $\mathcal{N}_\pm$. Equation (4.16) implies the same statement about the operator valued function $QR(z)Q^*$. We also take into account that $QR(z)Q^*$ and $QR(\bar{z})Q^*$ are continuous simultaneously. Thus, combining Proposition 2.5 and Lemma 4.9, we obtain the following assertion, which is known as the limiting absorption principle.

Theorem 4.10. *Let Assumption* 4.1(ii − vii) *hold. Then the set* $\mathcal{N}_\pm \subset \Delta$ *is closed and has the Lebesgue measure zero. The operator-valued function* $QR(z)Q^*$ *is Hölder continuous with the exponent* γ *up to the cut along* Δ *away from the set* $\mathcal{N} = \mathcal{N}_+ \cap \mathcal{N}_-$.

Observe that the hypotheses of Theorem 4.10 do not exclude that, for example, $J = 0$; then $Q = 0$ and the statement of the theorem is vacuous. So in order to deduce from this theorem some spectral consequences for A, we need additional assumptions such as $\operatorname{Ker} J^* = \{0\}$ (which is part (i) of Assumption 4.1). Then the kernel of $Q = Q_0 J^*$ is trivial and hence the range of the operator Q^* is dense in $\mathcal{H}$. So Theorem 4.10 implies the following result.

Corollary 4.11. *Let Assumption* 4.1(i − vii) *hold. Then, on the interval* Δ, *the singular continuous spectrum and the eigenvalues of* A *are contained in the set* $\mathcal{N}$.

Next, we check that the "exceptional" set $\mathcal{N}$ is exhausted by the eigenvalues of the operator A. To that end, we need to study in more detail the solutions of the homogeneous equation (4.17). We start with an elementary but not quite obvious identity which is a direct consequence of the self-adjointness of A.

Lemma 4.12. *Let Assumption* 4.1(iii, vi, vii) *hold. For all* $g \in \mathcal{H}_0$ *and* $z = \lambda + i\varepsilon$, $\varepsilon \neq 0$, *we have*

$$\operatorname{Im}\left((I + G_0(z))g, MQ_0R_0(z)Q_0^*g\right) = -\varepsilon\|JR_0(z)Q_0^*g\|^2. \qquad (4.18)$$

Proof. By the relation (1.5), we have

$$J(A_0 - z)\varphi + JT\varphi = AJ\varphi - zJ\varphi$$

for all $\varphi \in \mathcal{H}_0$. Setting here $\varphi = R_0(z)f$, we see that

$$J(I + TR_0(z))f = AJR_0(z)f - zJR_0(z)f$$

and hence

$$(J(I + TR_0(z))f, JR_0(z)f) = (AJR_0(z)f, JR_0(z)f) - z(JR_0(z)f, JR_0(z)f).$$

Taking the imaginary part and using the self-adjointness of A, we get

$$\operatorname{Im}(J(I + TR_0(z))f, JR_0(z)f) = -\varepsilon\|JR_0(z)f\|^2.$$

Setting $f = Q_0^*g$, we obtain

$$\operatorname{Im}\left((I + TR_0(z))Q_0^*g, J^*JR_0(z)Q_0^*g\right) = -\varepsilon\|JR_0(z)Q_0^*g\|^2. \qquad (4.19)$$

According to identities (4.11) and (4.14) the l.h.s. of (4.18) and of (4.19) coincide. □

In the case $J = I$, identity (4.18) is well known and plays the crucial role in the study of the exceptional set $\mathcal{N}$. This is still true in a more general case considered here.

Our next goal is to pass to the limit $\varepsilon \to 0$ in (4.18). The following assertion will allow us to get rid of the operator J in the r.h.s.

Lemma 4.13. *Let Assumption* 4.1(iv, vii, viii) *hold. Then, for all $g \in \mathcal{H}_0$, the function*

$$((J^*J - I)R_0(z)Q_0^*g, R_0(z)Q_0^*g)$$

is continuous for $\operatorname{Re} z \in \Delta$, $\pm\operatorname{Im} z \geq 0$.

Proof. Using identity (4.12), we get

$$|z|^2 R_0(\overline{z})(J^*J - I)R_0(z)$$
$$= -(J^*J - I) + R_0(\overline{z})A_0(J^*J - I)A_0 R_0(z) - 2\operatorname{Re}\left(z(J^*J - I)R_0(z)\right).$$

Consider separately the three terms in the r.h.s. The first one does not depend on z. Next, by Assumption 4.1(viii), we have

$$(A_0(J^*J - I)A_0 R_0(z)Q_0^*g, R_0(z)Q_0^*g) = (\widetilde{K}Q_0 R_0(z)Q_0^*g, Q_0 R_0(z)Q_0^*g).$$

This function is continuous because, by Lemma 4.9, the operator valued function $Q_0 R_0(z)Q_0^*$ is continuous. Finally, we have

$$((J^*J - I)R_0(z)Q_0^*g, Q_0^*g) = (Q_0 J^* J R_0(z)Q_0^*g, g) - (Q_0 R_0(z)Q_0^*g, g)$$
$$= (M Q_0 R_0(z)Q_0^*g, g) - (Q_0 R_0(z)Q_0^*g, g),$$

where we have used (vii) at the last step. The r.h.s. here is again continuous because the operator valued function $Q_0 R_0(z)Q_0^*$ is continuous. $\square$

Lemma 4.14. *Let Assumption* 4.1(iii, iv, vi − viii) *hold. If g satisfies equation* (4.17) *for $\lambda \in \Delta$, then*

$$\frac{d(E_0(-\infty, \lambda)Q_0^*g, Q_0^*g)}{d\lambda} = 0. \tag{4.20}$$

Proof. Recall that according to relation (2.10) under assumption (iv) the l.h.s. of (4.20) is a continuous function of $\lambda \in \Delta$. By Lemma 4.9, the operator valued function $G_0(z)$ defined by (4.9) is continuous for $\operatorname{Re} z \in \Delta$, $\pm\operatorname{Im} z \geq 0$. Therefore if g satisfies equation (4.17), then $\|(I + G_0(\lambda \pm i\varepsilon))g\| \to 0$ as $\varepsilon \to +0$. By Lemma 4.12, it follows that

$$\lim_{\varepsilon \to +0} \varepsilon \|J R_0(\lambda \pm i\varepsilon)Q_0^*g\|^2 = 0,$$

whence, by Lemma 4.13,

$$\lim_{\varepsilon \to +0} \varepsilon \|R_0(\lambda \pm i\varepsilon)Q_0^*g\|^2 = 0. \tag{4.21}$$

Now it remains to use the general operator theoretic identity

$$\frac{d(E_0(-\infty, \lambda)Q_0^*g, Q_0^*g)}{d\lambda} = \frac{1}{\pi} \lim_{\varepsilon \to +0} \varepsilon \|R_0(\lambda \pm i\varepsilon)Q_0^*g\|^2, \qquad (4.22)$$

which is a consequence of the relation between boundary values of a Cauchy integral and its density. Putting (4.21) and (4.22) together, we get (4.20). □

Given identity (4.20), the following two lemmas as well as the results of Subsection 4.5 are quite standard.

Lemma 4.15. *Let Assumption* $4.1(\text{ii} - \text{viii})$ *hold. Then for both signs* " $\pm$ " *the inclusion*

$$\mathcal{N}_\pm \subset \sigma_p(A) \cap \Delta \qquad (4.23)$$

is true.

Proof. Let a vector $g \neq 0$ satisfy equation (4.17). Set $f = Q_0^*g$ and $\varphi = R_0(\lambda \pm i0)f$; let us check that $\varphi \in \mathcal{H}_0$. Let F_0 be the unitary map (see (2.6)) which diagonalises A_0 and $\widehat{f} = F_0 f$. By assumption (iv) (the strong smoothness of Q_0), the function $\widehat{f}(\lambda) = Z_0(\lambda; Q_0)g$ defined by (2.9) is Hölder continuous on Δ with the exponent $\gamma > 1/2$. In view of equality (2.10) and Lemma 4.14, we have $\widehat{f}(\lambda) = 0$. Therefore

$$\|E_0(\Delta)\varphi\|^2 = \int_\Delta |\mu - \lambda|^{-2} |\widehat{f}(\mu)|^2 d\mu$$

$$= \int_\Delta |\mu - \lambda|^{-2} |\widehat{f}(\mu) - \widehat{f}(\lambda)|^2 d\mu \leq \text{const} \int_\Delta |\mu - \lambda|^{-2+2\gamma} d\mu < \infty,$$

and hence $\varphi \in \mathcal{H}_0$.

The following argument is quite similar to the proof of Lemma 4.7. Multiplying (4.17) by Q_0^* and using identity (4.11), we obtain

$$f + TR_0(\lambda \pm i0)f = 0.$$

It follows that $\varphi = R_0(\lambda \pm i0)f$ satisfies

$$(A_0 - \lambda)\varphi + T\varphi = 0. \qquad (4.24)$$

Since $AJ = J(A_0 + T)$, this yields $AJ\varphi = \lambda J\varphi$. So it remains to check that $J\varphi \neq 0$. Supposing the contrary and using equation (4.24), we see that $\varphi = 0$, by assumption (ii). Now it follows that $f = Q_0^*g = 0$ and hence $g = 0$. This contradicts the assumption $g \neq 0$. Thus $\psi = J\varphi \neq 0$ and $A\psi = \lambda\psi$. □

Lemma 4.16. *Let Assumption* $4.1(\text{i} - \text{viii})$ *hold true. Then on the interval* Δ, *the operator* A *does not have any singular continuous spectrum. For the point spectrum, we have*

$$\mathcal{N}_+ = \mathcal{N}_- = \sigma_p(A) \cap \Delta. \qquad (4.25)$$

Proof. By Corollary 4.11, the singular spectrum of A on Δ is contained in $\mathcal{N}$ and, in particular,

$$\sigma_p(A) \cap \Delta \subset \mathcal{N}. \tag{4.26}$$

Since, by Lemma 4.15, the set $\mathcal{N}$ is countable, the singular continuous spectrum of A is empty. Comparing (4.23) with (4.26), we obtain equality (4.25). □

4.4. **Non-accumulation of eigenvalues.** Here we prove two results.

Lemma 4.17. *Let Assumption* $4.1(\mathrm{i} - \mathrm{viii})$ *hold true. Then the eigenvalues of A in Δ have finite multiplicities.*

Proof. Taking conjugates in (1.5), we see that

$$J^*A - A_0J^* = T^*J^*.$$

Therefore if $A\psi = \lambda\psi$, then the element $\varphi = J^*\psi$ satisfies the equation

$$(A_0 - \lambda)\varphi + T^*\varphi = 0$$

and hence the equation

$$\varphi + R_0(\lambda \pm i0)T^*\varphi = 0.$$

Let us apply Q_0 to the last equation and use the fact that according to (4.11) $Q_0R_0(\lambda \pm i0)T^* = G_0(\lambda \pm i0)Q_0$. Thus, for $g = Q_0\varphi$, we get the equation

$$g + G_0^*(\lambda \pm i0)g = 0.$$

Next, we claim that $g \neq 0$ if $\psi \neq 0$. Indeed, if $g = 0$, then $\varphi = 0$ because $\operatorname{Ker} Q_0 = \{0\}$ and $\psi = 0$ because $\operatorname{Ker} J^* = \{0\}$. Actually, the above argument shows that

$$\dim \operatorname{Ker}(A - \lambda) \leq \dim \operatorname{Ker}(I + G_0^*(\lambda \pm i0)).$$

The dimension in the r.h.s. is finite because the operator $G_0^*(\lambda \pm i0)^2$ is compact. □

Lemma 4.18. *Let Assumption* $4.1(\mathrm{i} - \mathrm{viii})$ *hold true. Then the eigenvalues of A in Δ can accumulate only to the endpoints of Δ.*

Proof. Suppose, to get a contradiction, that a sequence of eigenvalues of A in Δ has an accumulation point: $\lambda_n \to \lambda_0 \in \Delta$ as $n \to \infty$. Then by Lemma 4.16 there exists a sequence of elements $g_n \in \mathcal{H}_0$ such that

$$g_n + G_0(\lambda_n + i0)g_n = 0, \quad \|g_n\| = 1. \tag{4.27}$$

Since the operators $G_0(\lambda + i0)$ depend continuously on $\lambda \in \Delta$ and $G_0(\lambda + i0)^2$ are compact, we may assume that

$$\|g_n - g_0\| \to 0, \quad n \to \infty, \tag{4.28}$$

where the element $g_0 \in \mathcal{H}_0$ satisfies

$$g_0 + G_0(\lambda_0 + i0)g_0 = 0, \quad \|g_0\| = 1.$$

Let us set

$$\psi_n = JR_0(\lambda_n + i0)Q_0^*g_n, \quad \psi_0 = JR_0(\lambda_0 + i0)Q_0^*g_0.$$

By the arguments of Lemma 4.15, it can be easily deduced from condition (4.20) on $Q_0^*g_n$ and $Q_0^*g_0$ that $\psi_n \in \mathcal{H}_0$ and $\psi_0 \in \mathcal{H}_0$. Using additionally (4.28), we obtain

$$\|\psi_n - \psi_0\| \to 0, \quad n \to \infty. \tag{4.29}$$

Exactly as in Lemma 4.15, equation (4.27) implies that $A\psi_n = \lambda_n\psi_n$, and hence ψ_n are pairwise orthogonal. Therefore relation (4.29) can be true only if $\psi_0 = 0$. Now, again the arguments of Lemma 4.15 show that $g_0 = 0$ which contradicts the condition $\|g_0\| = 1$. $\qquad\square$

Combining Theorem 4.10 and Lemmas 4.16 – 4.18, we obtain statements (ii), (iii) and (iv) of Theorem 4.2.

4.5. **The wave operators.** Here we prove Theorem 4.3. Let us set $\widetilde{J} = JA_0$ and first prove intermediate results involving the identification $\widetilde{J}$ instead of J.

Lemma 4.19. *Let Assumption* 4.1 *hold true. Then the wave operators*

$$W_\pm(A, A_0; \widetilde{J}, \Delta), \quad W_\pm(A_0, A; \widetilde{J}^*, \Delta) \tag{4.30}$$

exist and satisfy the relations

$$W_\pm^*(A, A_0; \widetilde{J}, \Delta) = W_\pm(A_0, A; \widetilde{J}^*, \Delta), \tag{4.31}$$

$$W_\pm^*(A, A_0; \widetilde{J}, \Delta)W_\pm(A, A_0; \widetilde{J}, \Delta) = A_0^2 E_0(\Delta), \tag{4.32}$$

$$W_\pm(A, A_0; \widetilde{J}, \Delta)W_\pm^*(A, A_0; \widetilde{J}, \Delta) = A^2 E^{(\mathrm{ac})}(\Delta). \tag{4.33}$$

Proof. By condition (1.5) and Assumption 4.1(v), we have

$$A\widetilde{J} - \widetilde{J}A_0 = JTA_0 = Q^*KQ_0, \quad Q = Q_0J^*.$$

By Assumption 4.1(iv), the operator Q_0 is strongly A_0-smooth on Δ and therefore it is A_0-smooth (in the sense of Kato) on any compact subinterval of Δ. Next, by Theorem 4.10, the operator Q is A-smooth (in the sense of Kato) on every compact subinterval of the set $\Delta \setminus \mathcal{N}$. Therefore Proposition 2.2 implies the existence of the wave operators (4.30); then relation (4.31) automatically holds.

By Assumptions 4.1(viii, ix), we have

$$\widetilde{J}^*\widetilde{J} - A_0^2 \in \mathfrak{S}_\infty, \quad \widetilde{J}\widetilde{J}^* - A^2 \in \mathfrak{S}_\infty.$$

Applying now Lemma 2.1 with $\varphi(\lambda) = \lambda^2$ to the triple $A_0, A, \widetilde{J}$, we obtain relation (4.32). Similarly, applying Lemma 2.1 to the triple $A, A_0, \widetilde{J}^*$, we obtain the relation

$$W_\pm^*(A_0, A; \widetilde{J}^*, \Delta)W_\pm(A_0, A; \widetilde{J}^*, \Delta) = A^2 E^{(\mathrm{ac})}(\Delta).$$

By (4.31), it is equivalent to (4.33). $\qquad\square$

Now we are ready to provide

Proof of Theorem 4.3. Since, by Lemma 4.19, the wave operators $W_\pm(A, A_0; \widetilde{J}, \Delta)$ exist, the limits (2.1) exist on elements $f = A_0 g$ where $g \in \mathcal{H}_0$ is arbitrary. Using that $\mathcal{H}_0^{(\mathrm{ac})} \subset \overline{\operatorname{Ran} A_0}$, we see that the wave operators $W_\pm(A, A_0; J, \Delta)$ also exist and satisfy

$$W_\pm(A, A_0; J, \Delta)A_0 = W_\pm(A, A_0; \widetilde{J}, \Delta). \tag{4.34}$$

It follows from (4.32) and (4.34) that for all $f \in \mathcal{H}_0$

$$\|W_\pm(A, A_0; J, \Delta)A_0 f\| = \|W_\pm(A, A_0; \widetilde{J}, \Delta)f\| = \|E_0(\Delta)A_0 f\|.$$

Therefore

$$\|W_\pm(A, A_0; J, \Delta)g\| = \|E_0(\Delta)g\| \tag{4.35}$$

for all $g \in \operatorname{Ran} A_0$ and hence for all $g \in \overline{\operatorname{Ran} A_0}$. Every $g \in \mathcal{H}_0$ equals $g = g_0 + g_1$ where $g_0 \in \operatorname{Ker} A_0$ and $g_1 \in \overline{\operatorname{Ran} A_0}$. Since $W_\pm(A, A_0; J, \Delta)g = W_\pm(A, A_0; J, \Delta)g_1$ and $E_0(\Delta)g = E_0(\Delta)g_1$, equality (4.35) extends to all $g \in \mathcal{H}_0$. This implies (4.2).

Next, by the intertwining relation, (4.34) yields

$$AW_\pm(A, A_0; J, \Delta) = W_\pm(A, A_0; \widetilde{J}, \Delta). \tag{4.36}$$

Similarly, to the proof of (4.35), comparing (4.33) and (4.36) , we see that

$$\|W_\pm^*(A, A_0; J, \Delta)g\| = \|E^{(\mathrm{ac})}(\Delta)g\|$$

for all $g \in \operatorname{Ran} A$. Then, again as (4.35), this equality extends to all $g \in \mathcal{H}$ which implies (4.3). $\qquad\square$

Finally, the first statement of Theorem 4.2 is a direct consequence of Theorem 4.3. The proofs of Theorems 4.2 and 4.3 are complete.

5. Proofs of Theorems 3.3 and 3.4

In this section we return to the setup of Section 3 and use Theorems 4.2 and 4.3 to prove our main results, Theorems 3.3 and 3.4, respectively.

Let $\mathcal{H}_0 = \mathcal{H}^{N+1}$, and let A, A_0, J be given by (1.1), (1.2), (1.4), respectively. We set $Q_0 = \operatorname{diag}\{X, \ldots, X\}$ in $\mathcal{H}_0$. As usual, Δ is a bounded open interval such that $0 \notin \Delta$.

Lemma 5.1. *Let Assumption* 3.2 *be satisfied. Then Assumption* 4.1 *holds true for the operators A_0, A and J defined above.*

Proof. (i) is obvious because

$$J^* f = (f, f, \ldots, f)^\top. \tag{5.1}$$

(ii) According to equality (1.6)

$$A_0 + T = \begin{pmatrix} A_\infty & A_\infty & \ldots & A_\infty \\ A_1 & A_1 & \ldots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ A_N & A_N & \ldots & A_N \end{pmatrix}$$

so that for $\mathbf{f} = (f_\infty, f_1, \ldots, f_N)^\top \in \mathcal{H}_0$ we have

$$(A_0 + T)\mathbf{f} = (A_\infty J\mathbf{f}, A_1 J\mathbf{f}, \ldots, A_N J\mathbf{f})^\top.$$

Thus if $J\mathbf{f} = 0$, then $(A_0 + T)\mathbf{f} = 0$. Hence $\mathbf{f} = 0$ if $J\mathbf{f} = (A_0 + T - z)\mathbf{f} = 0$.

(iii) follows from Assumption 3.2(i).

(iv) follows from Assumption 3.2(ii).

(v) According to (1.6) the operator TA_0 has matrix entries $A_\infty A_\ell$ and $A_j A_\ell$, $j \neq \ell$. It follows from Assumption 3.2(iii, v) that $A_\infty A_\ell = X^* K_\infty (X A_\ell X^{-1}) X$ where the operators $K_\infty (X A_\ell X^{-1})$ are compact. For $A_j A_\ell$, we have representation (3.3) where $K_{j,\ell}$ are compact.

(vi) According to (1.6) the operator $M_0 = Q_0 T^* Q_0^{-1}$ in $\mathcal{H}_0$ is represented by a matrix with entries $X A_\infty X^{-1}$ (which equals $X^* X K_\infty$ and is compact by Assumption 3.2(iii)) and $X A_j X^{-1}$ (which are bounded by Assumption 3.2(v)).

Similarly, the operator $M_0^2 = Q_0 (T^*)^2 Q_0^{-1}$ is represented by a matrix with entries $X A_j A_\ell X^{-1}$, $j \neq \ell$, $X A_\infty A_j X^{-1} = (X A_\infty X^{-1})(X A_j X^{-1})$, $X A_j A_\infty X^{-1} = (X A_j X^{-1})(X A_\infty X^{-1})$ and $X A_\infty^2 X^{-1} = (X A_\infty X^{-1})^2$. The operators $X A_j A_\ell X^{-1} = X X^* K_{j,\ell}$, $j \neq \ell$, are compact by Assumption 3.2(iv). As we have already seen, the other operators above are compact by Assumption 3.2(iii,v).

(vii) It follows from formulas (1.4) and (5.1) that the operator $J^* J$ acting in the space $\mathcal{H}_0$ has the form

$$J^* J = \begin{pmatrix} I & I & \ldots & I \\ I & I & \ldots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \ldots & I \end{pmatrix}. \tag{5.2}$$

Thus $Q_0 J^* J Q_0^{-1} = J^* J$ is a bounded operator.

(viii) According to formula (5.2) the operator $A_0 (J^* J - I) A_0$ in $\mathcal{H}_0$ has matrix entries which are zero on the diagonal and are of the form $A_j A_\ell$ with $j \neq$

ℓ off the diagonal. These operators admit representation (3.3) with compact operators $K_{j,\ell}$.

(ix) It follows from definitions (1.1), (1.2) and (1.4) that

$$JA_0^2 J^* - A^2 = \sum_{j=1}^{N} A_j^2 - (A_\infty + \sum_{j=1}^{N} A_j)^2.$$

In this expression the operators A_j^2 cancel each other. Therefore $JA_0^2 J^* - A^2$ consists of the terms A_∞^2, $A_\infty A_j$, $A_j A_\infty$ and $A_j A_\ell$, $j \neq \ell$, which are all compact by Assumption 3.2 (iii, iv). $\qquad\square$

Thus Theorems 4.2 and 4.3 are true for the operators A_0, A and J considered here. Theorem 3.3 is a direct consequence of Theorem 4.2. It remains to reformulate Theorem 4.3 as Theorem 3.4. By the definition (1.4) of J, the existence of the wave operators $W_\pm(A, A_0; J, \Delta)$ and the existence of $W_\pm(A, A_j; \Delta)$ for all $j = 1, \ldots, N$ are equivalent and

$$W_\pm(A, A_0; J, \Delta)\mathbf{f} = \sum_{j=1}^{N} W_\pm(A, A_j; \Delta) f_j \qquad (5.3)$$

if $\mathbf{f} = (f_0, f_1, \ldots, f_N)^\top$. The isometricity of $W_\pm(A, A_j; \Delta)$ and the intertwining property are the consequences of their existence.

Next, taking $\mathbf{f} = (0, \ldots, 0, f_j, 0, \ldots, 0)^\top$, $\mathbf{g} = (0, \ldots, 0, g_\ell, 0, \ldots, 0)^\top$ and using (4.2), (5.3), we obtain that

$$(W_\pm(A, A_j; \Delta) f_j, W_\pm(A, A_\ell; \Delta) g_\ell) = (E_0(\Delta)\mathbf{f}, \mathbf{g}).$$

If $j \neq \ell$, the r.h.s. here is zero for arbitrary $f_j \in \mathcal{H}$, $g_\ell \in \mathcal{H}$ which implies relation (3.4).

According to equality (4.3) for every $g \in \operatorname{Ran} E^{(\mathrm{ac})}(\Delta)$ and $\mathbf{f} = W_\pm^*(A, A_0; J, \Delta)g$, we have $g = W_\pm(A, A_0; J, \Delta)\mathbf{f}$. Therefore, again by (5.3), $g = \sum_{j=1}^{N} W_\pm(A, A_j; \Delta) f_j$ where $\mathbf{f} = (0, f_1, \ldots, f_N)^\top$. This proves the asymptotic completeness (3.5). $\qquad\square$

6. Stationary representations for wave operators and scattering matrix

Here we address a more special question of stationary representations for the wave operators and scattering matrix in the "abstract" framework of Section 4. Of course the representations obtained are automatically true for the triple (1.1), (1.2), (1.4). This is briefly discussed in Subsection 6.4.

6.1. **The scattering matrix: definition.** The local scattering operator for the triple A_0, A, J and the interval Δ is defined by the formula

$$\mathbf{S}(A, A_0; J, \Delta) = W_+(A, A_0; J, \Delta)^* W_-(A, A_0; J, \Delta). \tag{6.1}$$

By (4.2), (4.3) and the intertwining property of the wave operators, the scattering operator is unitary on $\operatorname{Ran} E_0(\Delta)$ and commutes with A_0.

Therefore in the spectral representation (see (2.6) and (2.7)) of A_0, the scattering operator acts as the multiplication by the operator valued function

$$S(\lambda) = S(\lambda; A, A_0; J, \Delta) : \mathfrak{h}_0 \to \mathfrak{h}_0.$$

It means that

$$(F_0 \mathbf{S}(A, A_0; J, \Delta) f)(\lambda) = S(\lambda)(F_0 f)(\lambda), \quad f \in \operatorname{Ran} E_0(\Delta).$$

The operator $S(\lambda)$ is defined and is unitary for almost all $\lambda \in \Delta$. It is known as the scattering matrix. The definition of the scattering matrix depends of course on the choice of the mapping (2.6), but in applications the mapping F_0 emerges naturally.

Along with the scattering matrix $S(\lambda)$ corresponding to the scattering operator (6.1), we consider the scattering matrix $\widetilde{S}(\lambda)$, corresponding to the scattering operator $\mathbf{S}(A, A_0; \widetilde{J}, \Delta)$ where $\widetilde{J} = JA_0$. Since

$$\begin{aligned}
\mathbf{S}(A, A_0; \widetilde{J}, \Delta) &= W_+(A, A_0; \widetilde{J}, \Delta)^* W_-(A, A_0; \widetilde{J}, \Delta) \\
&= A_0 W_+(A, A_0; J, \Delta)^* W_-(A, A_0; J, \Delta) A_0 = A_0 \mathbf{S}(A, A_0; J, \Delta) A_0,
\end{aligned}$$

we have

$$\widetilde{S}(\lambda) = \lambda^2 S(\lambda). \tag{6.2}$$

Note that the operators $\widetilde{S}(\lambda)$ are not unitary.

6.2. **The stationary representaton for the scattering matrix.** Our goal here is to obtain a representaton for the scattering matrix $S(\lambda)$ in terms of the resolvent $R(z) = (A - z)^{-1}$ of the operator A. As before, we set $Q = Q_0 J^* : \mathcal{H} \to \mathcal{H}_0$. Recall that, by Theorem 4.2, the operator valued function $G(z) = QR(z)Q^*$ is Hölder continuous (in the operator norm) in z for $\pm \operatorname{Im} z \geq 0$ and $\operatorname{Re} z \in \Delta \setminus \sigma_p(A)$. We also use the notation $Z_0(\lambda) = Z_0(\lambda; Q_0)$ for operator (2.9). This operator is bounded and depends Hölder continuously on $\lambda \in \Delta$. Now we are ready to present the stationary representation of $S(\lambda)$.

Theorem 6.1. *Let Assumption 4.1 hold. Then for all $\lambda \in \Delta \setminus \sigma_p(A)$ the scattering matrix $S(\lambda)$ can be represented as*

$$S(\lambda) = I - 2\pi i \lambda^{-1} Z_0(\lambda) M^* K Z_0^*(\lambda) + 2\pi i \lambda^{-2} Z_0(\lambda) K^* G(\lambda + i0) K Z_0^*(\lambda). \tag{6.3}$$

The operator $S(\lambda) - I$ is compact and depends Hölder continuously on λ.

We emphasize that all operators in the r.h.s. of (6.3) are bounded and depend Hölder continuously on $\lambda \in \Delta \setminus \sigma_p(A)$.

For the proof, we use the fact that all the assumptions of the general stationary scheme (see [1, 11]) are satisfied for the triple A_0, A, $\widetilde{J}$ (but not for A_0, A, J). Therefore we can apply the standard stationary representation for $\widetilde{S}(\lambda)$ (see Proposition 1 of § 7.4 in [11]). We recall this representation at a somewhat heuristic level. It is convenient to use a formal notation $\Gamma_0(\lambda) : \mathcal{H}_0 \to \mathfrak{h}_0$ defined by the equality

$$\Gamma_0(\lambda)f_0 = (F_0 f)(\lambda), \quad f \in \operatorname{Ran} E_0(\Delta), \quad \lambda \in \Delta. \tag{6.4}$$

Observe that the operator

$$Z_0(\lambda) = \Gamma_0(\lambda)Q_0^*$$

is correctly defined by equality (2.9). In view of (1.5) we have

$$\widetilde{V} := A\widetilde{J} - \widetilde{J}A_0 = JTA_0. \tag{6.5}$$

We further observe that the auxiliary wave operator $\Omega = W_+(A_0, A_0; \widetilde{J}^*\widetilde{J}, \Delta)$ exists. It commutes with the operator A_0 and hence acts as the multiplication by the operator valued function $\Omega(\lambda)$ in the spectral representation of A_0. Then the representation for $\widetilde{S}(\lambda)$ formally reads as

$$\widetilde{S}(\lambda) = \Omega(\lambda) - 2\pi i \Gamma_0(\lambda)(\widetilde{J}^*\widetilde{V} - \widetilde{V}^* R(\lambda + i0)\widetilde{V})\Gamma_0^*(\lambda).$$

Let us show that all the terms in the r.h.s. are correctly defined. First, we observe that under Assumption 4.1(viii) $\widetilde{J}^*\widetilde{J} - A_0^2 \in \mathfrak{S}_\infty$ so that the operator Ω exists and $\Omega = A_0^2 E_0(\Delta)$. It follows that $\Omega(\lambda) = \lambda^2 I$. Then we use the fact that according to (6.5) and Assumption 4.1(v, vii)

$$\widetilde{J}^*\widetilde{V} = A_0 J^* JTA_0 = A_0 Q_0^* M^* KQ_0$$

whence

$$\Gamma_0(\lambda)\widetilde{J}^*\widetilde{V}\Gamma_0^*(\lambda) = \lambda Z_0(\lambda)M^* KZ_0^*(\lambda).$$

Finally, according to (6.5) and Assumption 4.1(v) we have $\widetilde{V} = Q^* KQ_0^*$ so that

$$\Gamma_0(\lambda)\widetilde{V}^* R(\lambda + i0)\widetilde{V}\Gamma_0^*(\lambda) = Z_0(\lambda)K^* G(\lambda + i0)KZ_0^*(\lambda).$$

Now we are in the position to formulate the precise result.

Lemma 6.2. *Let Assumption* 4.1 *hold. Then for all* $\lambda \in \Delta \setminus \sigma_p(A)$ *the scattering matrix* $\widetilde{S}(\lambda)$ *can be represented as*

$$\widetilde{S}(\lambda) = \lambda^2 I - 2\pi i \lambda Z_0(\lambda)M^* KZ_0^*(\lambda) + 2\pi i Z_0(\lambda)K^* G(\lambda + i0)KZ_0^*(\lambda). \tag{6.6}$$

The representation (6.3) for $S(\lambda)$ directly follows from (6.2) and (6.6). Since $Z_0(\lambda)$ and $G(\lambda + i0)$ depend Hölder continuously on λ, the same is true for $S(\lambda)$. Finally, the operator $S(\lambda) - I$ is compact because by Assumption 4.1(v) the operator K is compact.

6.3. **Wave operators.** Here we briefly discuss stationary representations of the wave operators. These representations are equivalent to the expansion over appropriate generalised eigenfunctions of the operator A.

Since all the assumptions of the stationary scheme of scattering theory are satisfied for the triple A, A_0, $\widetilde{J}$, we can directly apply Theorem 5.6.1 of [11] to this triple. We use notation (6.4) and formally set

$$\widetilde{\Gamma}_\pm(\lambda)f = \Gamma_0(\lambda)(\widetilde{J}^* - \widetilde{V}^*R(\lambda \pm i0))f \tag{6.7}$$

for $f \in \operatorname{Ran} Q^*$. Similarly to the previous subsection, under Assumption 4.1 this formula acquires the correct meaning. Indeed, Assumption 4.1(vii) shows that

$$\Gamma_0(\lambda)\widetilde{J}^*Q^* = \lambda\Gamma_0(\lambda)J^*JQ_0^* = \lambda\Gamma_0(\lambda)Q_0^*M^* = \lambda Z_0(\lambda)M^*$$

and, in view of equality (6.5), Assumption 4.1(v) shows that

$$\Gamma_0(\lambda)\widetilde{V}^*R(\lambda \pm i0)Q^* = \Gamma_0(\lambda)A_0T^*J^*R(\lambda \pm i0)Q^*$$
$$= \Gamma_0(\lambda)Q_0^*K^*Q_0J^*R(\lambda \pm i0)Q^* = Z_0(\lambda)K^*G(\lambda \pm i0).$$

Therefore representation (6.7) can be rewritten in terms of bounded operators as

$$\widetilde{\Gamma}_\pm(\lambda)f = \lambda Z_0(\lambda)M^*g - Z_0(\lambda)K^*G(\lambda \pm i0)g, \quad f = Q^*g. \tag{6.8}$$

Now we put

$$(\widetilde{F}_\pm f)(\lambda) = \widetilde{\Gamma}_\pm(\lambda)f, \quad f \in \operatorname{Ran} Q^*.$$

According to Theorem 5.6.1 of [11] the operators $\widetilde{F}_\pm$ extend to bounded operators from $\mathcal{H}$ to $L^2(\Delta, \mathfrak{h}_0)$ and diagonalize the operator A: $(\widetilde{F}_\pm Af)(\lambda) = \lambda(\widetilde{F}_\pm f)(\lambda)$. They are related to the wave operators by the formula $W_\pm(A, A_0; \widetilde{J}, \Delta) = \widetilde{F}_\pm^* F_0$.

It remains to replace the identification $\widetilde{J}$ by J. We again formally set

$$\Gamma_\pm(\lambda)f = \Gamma_0(\lambda)(J^* - T^*J^*R(\lambda \pm i0))f \tag{6.9}$$

for $f \in \operatorname{Ran} Q^*$. Since $\widetilde{J} = JA_0$ and $\widetilde{V} = JTA_0$, comparing (6.7) and (6.9) we see that $\widetilde{\Gamma}_\pm(\lambda) = \lambda\Gamma_\pm(\lambda)$. Therefore using (6.8), we can rewrite (6.9) in terms of bounded operators as

$$\Gamma_\pm(\lambda)f = Z_0(\lambda)M^*g - \lambda^{-1}Z_0(\lambda)K^*G(\lambda \pm i0)g, \quad f = Q^*g. \tag{6.10}$$

Thus, the operators $\Gamma_\pm(\lambda) : \mathcal{H} \to \mathfrak{h}_0$ are well defined on the dense set $\operatorname{Ran} Q^*$ and, for $f \in \operatorname{Ran} Q^*$, the vector valued functions $\Gamma_\pm(\lambda)f$ depend Hölder continuously on $\lambda \in \Delta \setminus \sigma_p(A)$. Using (4.34), we can now rephrase the results

about the wave operators $W_\pm(A, A_0; \widetilde{J}, \Delta)$ in terms of the wave operators $W_\pm(A, A_0; J, \Delta)$. This yields the following result.

Theorem 6.3. *Let Assumption 4.1 hold. Define the operators $\Gamma_\pm(\lambda)$ by equation (6.9) (or, more precisely, by (6.10)) and set*

$$(F_\pm f)(\lambda) = \Gamma_\pm(\lambda)f, \quad f \in \operatorname{Ran} Q^*.$$

Then $F_\pm$ extends to the partial isometry $F_\pm : \mathcal{H} \to L^2(\Delta; \mathfrak{h}_0)$ with the initial space $\operatorname{Ran} E^{(\mathrm{ac})}(\Delta)$, and the intertwining property

$$(F_\pm A f)(\lambda) = \lambda(F_\pm f)(\lambda)$$

is satisfied. The operators $F_\pm$ and the wave operators are related by the equality

$$W_\pm(A, A_0; J, \Delta) = F_\pm^* F_0.$$

Moreover, for $f_0 \in \operatorname{Ran} Q_0^*$, $f \in \operatorname{Ran} Q^*$, we have the representation

$$(W_\pm(A, A_0; J, \Delta)f_0, f) = \int_\Delta \langle \Gamma_0(\lambda)f_0, \Gamma_\pm(\lambda)f \rangle d\lambda$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathfrak{h}_0$.

6.4. **The multichannel case.** Of course under Assumption 3.2 all the results of this section are true for the operators A_0, A and J defined by (1.1), (1.2) and (1.4). Let A_j, $j = 1, \ldots, N$, be realized as the operator of multiplication by independent variable λ in the space $L^2(\Delta) \otimes \mathfrak{h}_j$ where $\dim \mathfrak{h}_j$ is the multiplicity of the spectrum of the operator A_j on the interval Δ. Then the scattering matrix $S(\lambda)$ is given by the matrix $S_{\ell,j}(\lambda) : \mathfrak{h}_j \to \mathfrak{h}_k$. Theorem 6.1, in particular, shows that the operators $S_{j,j}(\lambda) - I$ and $S_{\ell,j}(\lambda)$ for $\ell \neq j$ are compact.

Appendix A. Faddeev's equations

Let us show that Faddeev's equations for three interacting quantum particles follow from the resolvent equation (1.8) for a particular choice of the operators A_0, A and J. Actually, we consider a slightly more general situation.

Let a self-adjoint operator H in a Hilbert space $\mathcal{H}$ admit the representation

$$H = H_0 + \sum_{j=1}^{N} V_j. \tag{A.1}$$

We suppose that the operator H_0 is self-adjoint and set $\mathcal{R}_0(z) = (H_0 - z)^{-1}$. For simplicity, we assume that all operators V_j (but not H_0) are bounded. Our main assumption is that

$$V_j \mathcal{R}_0(z) V_k \in \mathfrak{S}_\infty, \quad j, k = 1, \ldots, N, \ j \neq k, \quad \operatorname{Im} z \neq 0. \tag{A.2}$$

In the three-particle problem H is the Schrödinger operator, H_0 is the operator of the kinetic energy of three particles with the center-of-mass motion removed; V_j, $j = 1, 2, 3$, are potential energies of pair interactions of particles (for example, V_1 is the potential energy of interaction of the second and third particles).

We introduce the Hilbert space $\mathcal{H}_0 = \mathcal{H}^N$ as the direct sum of N copies of the space $\mathcal{H}$. The elements of this space are columns $\mathbf{f} = (f_1, \ldots, f_N)^\top$. We define the operator A_0 in this space as

$$A_0 = \mathrm{diag}\{H_1, \ldots, H_N\} \quad \text{where} \quad H_j = H_0 + V_j \tag{A.3}$$

and the operator $J : \mathcal{H}_0 \to \mathcal{H}$ by

$$J\mathbf{f} = \sum_{j=1}^{N} f_j. \tag{A.4}$$

We set $A = H$.

Since

$$(AJ - JA_0)\mathbf{f} = \sum_{j,k=1; j \neq k}^{N} V_k f_j$$

factorization (1.5) is now true with the operator $T : \mathcal{H}_0 \to \mathcal{H}_0$ acting by the formula

$$T = \begin{pmatrix} 0 & V_1 & V_1 & \cdots & V_1 \\ V_2 & 0 & V_2 & \cdots & V_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_N & V_N & V_N & \cdots & 0 \end{pmatrix}. \tag{A.5}$$

Let us check Assumption 4.1(ii). Comparing equations (A.3) and (A.5) we find that

$$A_0 + T = \begin{pmatrix} H_1 & V_1 & V_1 & \cdots & V_1 \\ V_2 & H_2 & V_2 & \cdots & V_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_N & V_N & V_N & \cdots & H_N \end{pmatrix}$$

and hence

$$(A_0 + T)\mathbf{f} = (H_0 f_1 + V_1 J\mathbf{f}, H_0 f_2 + V_2 J\mathbf{f}, \ldots, H_0 f_N + V_N J\mathbf{f})^\top.$$

Thus, if $J\mathbf{f} = 0$, then

$$(A_0 + T - z)\mathbf{f} = ((H_0 - z)f_1, (H_0 - z)f_2, \ldots, (H_0 - z)f_N)^\top.$$

Since the operator H_0 is self-adjoint, the equality $(A_0 + T - z)\mathbf{f} = 0$ implies that $f_j = 0$ for all $j = 1, 2, \ldots, N$.

Next, we check inclusion (4.4) for $p = 2$. Set $\mathcal{R}_j(z) = (H_j - z)^{-1}$. It follows from (A.5) that

$$
TR_0(z) = \begin{pmatrix} 0 & V_1\mathcal{R}_2(z) & V_1\mathcal{R}_3(z) & \dots & V_1\mathcal{R}_N(z) \\ V_2\mathcal{R}_1(z) & 0 & V_2\mathcal{R}_3(z) & \dots & V_2\mathcal{R}_N(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ V_N\mathcal{R}_1(z) & V_N\mathcal{R}_2(z) & V_N\mathcal{R}_3(z) & \dots & 0 \end{pmatrix}.
$$

Therefore the operator $(TR_0(z))^2$ is given by the $N \times N$ matrix with elements

$$
V_j\mathcal{R}_k(z)V_k\mathcal{R}_\ell(z), \quad j \neq k.
$$

By the resolvent identity applied to the pair H_0, H_k, we have

$$
V_j\mathcal{R}_kV_k = (V_j\mathcal{R}_0V_k)(I - \mathcal{R}_kV_k),
$$

and hence this operator is compact by assumption (A.2).

Thus Lemma 4.5 implies the following result.

Theorem A.1. *Let the operators A_0 and $A = H$ be defined by formulas (A.3) and (A.1), and let the operator J be given by formula (A.4). Then under assumption (A.2) the resolvent $R(z) = (H - z)^{-1}$ of the operator H admits the representation (4.7) where the inverse operator on the right exists and is bounded.*

Note that equation (1.8) in the case considered is equivalent to the system of Faddeev's equations. Indeed, applying both sides of (1.8) to an element $\mathbf{f} = (f_1, \ldots, f_N)^\top$, we find that

$$
R(z)\Big(\sum_{k=1}^N f_k + \sum_{j,k=1;j\neq k}^N V_j\mathcal{R}_k(z)f_k \Big) = \sum_{k=1}^N \mathcal{R}_k(z)f_k.
$$

Since elements f_k are arbitrary, this leads to a system of N equations

$$
R(z)\Big(I + \sum_{j=1;j\neq k}^N V_j\mathcal{R}_k(z)\Big) = \mathcal{R}_k(z), \quad k = 1, \ldots, N, \tag{A.6}
$$

for the same object $R(z)$. We note that each of equations (A.6) is simply the resolvent equation for the pair H_k, H. It determines the resolvent uniquely, but the operators $V_j\mathcal{R}_k(z)$ are not of course compact. Nevertheless, considered together, equations (A.6) yield the Fredholm system.

Applying to (A.6) on the right the operator V_k and setting $Y_k(z) = R(z)V_k$, we obtain the system of N equations

$$
Y_k(z) + \sum_{j=1;j\neq k}^N Y_j(z)\mathcal{R}_k(z)V_k = \mathcal{R}_k(z)V_k, \quad k = 1, \ldots, N,
$$

for N operators $Y_k(z)$. This system was derived and used by L. D. Faddeev [2] in the study of three particle quantum systems.

Acknowledgements

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